Example: Find L.T. of $(2+3t)^2$

Sol. $L\{(2+3t)^2\} = L\{4+9t^2+12t\}$

$= L\{4\} + L\{9t^2\} + L\{12t\}$ (using Linear Property)

$= 4.L\{1\} + 9\,L\{t^2\} + 12\,L\{t\}$ (— do —)

$= 4\times\frac{1}{s} + 9.\frac{2!}{s^{2+1}} + 12\,\frac{1}{s^2}$

$= \frac{4}{s} + \frac{18}{s^3} + \frac{12}{s^2}$ Ans.

— x —

Ex. find L.T. $\cos(at+b)$

Sol: $L\{\cos(at+b)\} = L\{\cos at.\cos b - \sin at\,\sin b\}$

$= \cos b.\,L\{\cos at\} - \sin b\,L\{\sin at\}$ [Linear Prop.

$= \cos b.\frac{s}{s^2+a^2} - \sin b.\frac{a}{s^2+a^2}$

$= (s.\cos b - a\sin b)/(s^2+a^2)$ Ans.

— x —

Ex find ~~L~~ L.T. of . $\cosh^2 at$.

Sol. $L\{\cosh^2 at\} = L\{(\cosh at)^2\} = L\{(\frac{e^t+e^{-t}}{2})^2\}$

$= L\{\frac{1}{4}(e^{2t}+e^{-2t}+2)\}$

$= \frac{1}{4}L\{e^{2t}\} + \frac{1}{4}L\{e^{-2t}\} + \frac{2}{4}L\{1\}$

$= \frac{1}{4}\,\frac{1}{s-2} + \frac{1}{4}\,\frac{1}{s+2} + \frac{1}{2}.\frac{1}{s}$

$= \frac{s(s+2)+s(s-2)+2(s-2)(s+2)}{4.s(s^2-4)} = \frac{4s^2-8}{4s(s^2-4)} = \frac{s^2-2}{s(s^2-4)}$

Ex find L.T. of. $\sin^2 \omega t$

Sol. $L\{\sin^2 \omega t\} = L\{\frac{1-\cos 2\omega t}{2}\}$

$= \frac{1}{2} L\{1\} - \frac{1}{2} \cdot L\{\cos 2\omega t\}$

$= \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s}{s^2+4\omega^2} = \frac{s^2+4\omega^2 - s^2}{2s(s^2+4\omega^2)}$

$= \cancel{s^2-4\omega^2} = \frac{2\omega^2}{s(s^2+4\omega^2)}$. Ans.

Ex : find L.T. of. $t.e^{2t}$

Sol: $L\{te^{2t}\} = \int_0^\infty e^{-st} . t . e^{2t} dt$ [By definition]

$= \int_0^\infty e^{-(s-2)t} . t \, dt$ [Solve by Parts]

$= \left[t . \frac{e^{-(s-2)t}}{-(s-2)}\right]_0^\infty - \int_0^\infty \frac{e^{-(s-2)t}}{-(s-2)} \times 1 . dt$

$= \frac{-1}{(s-2)} \cdot \left[\frac{t}{e^{(s-2)t}}\right]_0^\infty + \frac{1}{s-2} \int_0^\infty e^{-(s-2)t} dt$

$= \frac{-1}{(s-2)} [0-0] + \frac{1}{s-2} \cdot \left[\frac{e^{-(s-2)t}}{-(s-2)}\right]_0^\infty$

$= -\frac{1}{(s-2)^2} [0-1] = \frac{1}{(s-2)^2}$. Ans.

Ex find L.T. of $f(t) = \begin{cases} \cos t & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$

Sol. Using definition of L.T.,

$$L\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_0^\pi e^{-st} \cdot f(t)\,dt + \int_\pi^\infty e^{-st} \cdot f(t)\,dt$$

$$= \int_0^\pi e^{-st} \cdot \cos t\, dt + \int_\pi^\infty e^{-st} \cdot 0 \cdot dt \quad \left[\text{Using the def. of } f(t)\right.$$

$$= \underbrace{\int_0^\pi e^{-st} \cdot \cos t\, dt}_{(1)} \quad (\text{Solve by Parts})$$

let

$$I = \int_0^\pi \cos t \cdot e^{-st}\, dt = \left[\cos t \cdot \frac{e^{-st}}{-s}\right]_0^\pi - \int_0^\pi \frac{e^{-st}}{-s} \cdot -\sin t\, dt$$

$$= -\frac{1}{s}\left[\frac{\cos t}{e^{st}}\right]_0^\pi - \frac{1}{s}\int_0^\pi e^{-st} \cdot \sin t\, dt$$

$$= \cancel{-\frac{1}{s}\left(\frac{\cos\pi}{e^{s\pi}} - \frac{1}{1}\right)} = -\frac{1}{s}\left[\frac{-1}{e^{s\pi}} - \frac{1}{1}\right] - \frac{1}{s}\left[\left(\sin t \cdot \frac{e^{-st}}{-s}\right)_0^\pi - \int_0^\pi \frac{e^{-st}}{-s} \times \cos t\, dt\right.$$

$$I = -\frac{1}{s}\left[-e^{-s\pi} - 1\right] - \frac{1}{s}\left[0 + \frac{1}{s} I\right)$$

$$= \frac{e^{-s\pi} + 1}{s} - \frac{I}{s^2}$$

$$I\left(1 + \frac{1}{s^2}\right) = \frac{e^{-s\pi} + 1}{s} \Rightarrow I = \frac{s(e^{-s\pi} + 1)}{s^2 + 1} \quad \text{Put in (1)}$$

$$\boxed{L\{f(t)\} = \frac{s(e^{-s\pi} + 1)}{s^2 + 1}} \quad \text{Ans.}$$

Ex find inverse Laplace Transform of $\frac{3}{s+5}$

Sol: $L^{-1}\left\{\frac{3}{s+5}\right\} = 3L^{-1}\left\{\frac{1}{s+5}\right\}$

$= 3L^{-1}\left\{\frac{1}{s-(-5)}\right\} =$ $\left[\text{Use } L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}\right.$

$= 3.\, e^{-5t}$ Ans.

Ex find inverse Laplace Transform of $\frac{s+3}{(s-1)(s+2)}$

Sol. Using Partial fraction, we can write

$$\frac{s+3}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2} \quad \text{--- (1)}$$

$$s+3 = A(s+2) + B(s-1)$$

Put $s=1$ $\quad 4 = A.3 + 0 \Rightarrow \boxed{A = \frac{4}{3}}$

Put $s=-2$ $\quad 1 = 0 + B(-3) \Rightarrow \boxed{B = -\frac{1}{3}}$ Put in (1)

$$\therefore \frac{s+3}{(s-1)(s+2)} = \frac{(4/3)}{s-1} + \frac{(-1/3)}{s+2}$$

$$\therefore L^{-1}\left\{\frac{s+3}{(s-1)(s+2)}\right\} = L^{-1}\left\{\frac{(4/3)}{s-1}\right\} + L^{-1}\left\{\frac{(-1/3)}{s+2}\right\}$$

$$= \frac{4}{3}L^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{4}{3}e^{t} - \frac{1}{3}e^{-2t}$$

$$L^{-1}\left\{\frac{s+3}{(s-1)(s+2)}\right\} = \frac{4e^{t} - e^{-2t}}{3}$$ Ans

Ex. Find Laplace Transform of $f(t) = t^{5/2}$ given that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Sol. We know that $L\{t^{\alpha}\} = \dfrac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $\alpha > 0$

Put $\alpha = 5/2$

$$\therefore L\{t^{5/2}\} = \frac{\Gamma(\frac{5}{2}+1)}{s^{5/2+1}} = \frac{\Gamma(\frac{7}{2})}{s^{7/2}}$$

$$= \frac{\frac{5}{2}\Gamma(\frac{5}{2})}{s^{7/2}} = \frac{\frac{5}{2}\Gamma(\frac{3}{2}+1)}{s^{7/2}} \quad [\text{using } \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma(\frac{3}{2})}{s^{7/2}} = \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma(\frac{1}{2}+1)}{s^{7/2}}$$

$$= \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(\frac{1}{2})}{s^{7/2}} = \frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{7/2}}$$

$$= \frac{15\cdot\sqrt{\pi}}{8\, s^{7/2}} \quad \text{-Ans}$$

Ex Show that $\displaystyle\int_{nT}^{(n+1)T} e^{-st}\cdot f(t)\,dt = e^{-nsT}\int_{0}^{T} e^{-st} f(t)\,dt$,

where n is an integer, when $f(t)$ is a periodic function of period T.

Solution: Since given function $f(t)$ is a periodic function of period T, therefore let

$t = u + nT$, when $t = nT \Rightarrow u = 0$

$t = (n+1)T \Rightarrow u = T$

$$\therefore \int_{nT}^{(n+1)T} e^{-st} f(t)\,dt = \int_0^T e^{-s(u+nT)} f(u+nT)\,du$$

$$= \int_0^T e^{-su} \cdot e^{-snT} \cdot f(u)\,du$$

$$= e^{-snT} \int_0^T e^{-su} f(u)\,du$$

$$= e^{-snT} \int_0^T e^{-st} f(t)\,dt \cdot \quad \text{Ans}$$

Assignment

Q. Find L.T. of following functions

1. $3t^2 - \cos 2t$ [Ans. $\dfrac{6s^2+24-s^4}{s^3(s^2+4)}$

2. e^{-2t+5} [Ans. $\dfrac{e^5}{s+2}$

3. $t \cdot \sin t$ [Ans. $\dfrac{2s}{(s^2+1)^2}$

4. $(\cos t + \sin t)^2$ [Ans. $\dfrac{1}{s} + \dfrac{2}{s^2+4}$

5. $f(t) = \begin{cases} 2 & 0 \le t < 3 \\ 0 & t \ge 3 \end{cases}$ [Ans $\dfrac{2(1-e^{-3s})}{s}$

6. $f(t) = \begin{cases} 0 & 0 \le t < \pi \\ \sin t & t \ge \pi \end{cases}$ [Ans. $\dfrac{-e^{-s\pi}}{s^2+1}$

7. $\sin 5t + \cos 4t$ $\left[\text{Ans } \frac{5}{s^2+25} + \frac{s}{s^2+16}\right.$

8. $e^{t} \cdot \sin t$ $\left[\text{Ans. } \frac{1}{(s-1)^2+1}\right.$

9. $e^{-t} \cdot \sinh t$ $\left[\text{Ans. } \frac{1}{s(s+2)}\right.$

Ex find inverse Laplace Transform of following

1. $\frac{2}{s^3} + \frac{6}{s^2} - \frac{5}{s}$ $\left[\text{Ans } t^2 + 6t - 5\right.$

2. $\frac{3}{s^2+2s}$ $\left[\text{Ans. } \frac{3(1-e^{-2t})}{2}\right.$

3. $\frac{s^2+2s+5}{(s-1)(s-2)(s-3)}$ $\left[\text{Ans. } 4e^{t} - 13e^{2t} + 10e^{3t}\right.$

Ex If $L\{f(t)\} = F(s)$ then show that

$$L\{f(at)\} = \frac{1}{a} \cdot F\left(\frac{s}{a}\right).$$

Ex find $L\{e^{(a+ib)t}\}$. Hence show that

$$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$

$$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}.$$

Laplace Transform to solve differential equations:-

We can use Laplace Transform to reduce differential equation to an algebraic equation which will be simpler than actual solving the differential equation directly in many cases. Laplace Transform can also be used to solve initial value problem. Before proceeding into differential equations we need formula to find the Laplace Transform of derivative.

Laplace Transform of Derivatives→

→ If $L(f(t)) = F(s)$, then $L(f'(t)) = sL(f(t)) - f(0) = sF(s) - f(0)$

By definition $L(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt \quad \text{[Integration by parts]}$$

$$\boxed{L(f'(t)) = -f(0) + sL(f(t))}$$

$$\boxed{L(f'(t)) = -f(0) + sF(s)} \quad \text{[Using (1)]} \quad \text{— (9)}$$

Since we are going to be dealing with higher order differential equation, so we will find Laplace transform of higher order derivatives.

Using (9) $L(f''(t)) = sL(f'(t)) - f'(0)$

$$= s[sL(f(t)) - f(0)] - f'(0)$$

$$\boxed{L(f''(t)) = s^2 L(f(t)) - sf(0) - f'(0)} \quad \text{— (10)}$$

$$L(f'''(t)) = sL(f''(t)) - f''(0)$$

$$= s[s^2 L(f(t)) - sf(0) - f'(0)] - f''(0)$$

$$\boxed{L(f'''(t)) = s^3 L(f(t)) - s^2 f(0) - sf'(0) - f''(0)} \quad \text{— (11)}$$

IIIly $L(f^n(t)) = s^n L(f(t)) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0)$

where $f^n(t)$ denotes the nth order derivative

Remark. Let $f(t)$ be function s.t $f(0) = 0$

Then using (9) $L(f'(t)) = sF(s)$

$$\therefore \boxed{L^{-1}(sF(s)) = f'(t) = \frac{d}{dt} L^{-1}(F(s))} \quad -(12)$$

eg Find $L^{-1}\left(\frac{s}{s^2+a^2}\right)$

Let $F(s) = \frac{1}{s^2+a^2}$ then $f(t) = L^{-1}(F(s))$

$$f(t) = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$$

Also $f(0) = 0$

$\therefore$ Using (12), $L^{-1}(sF(s)) = f'(t)$

$$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \frac{d}{dt}\left(\frac{\sin at}{a}\right) = \cos at$$

eg Given that $L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-t} - e^{-2t}$, find $L^{-1}\left[\frac{s}{(s+1)(s+2)}\right]$

Let $F(s) = \frac{1}{(s+1)(s+2)}$ then $f(t) = L^{-1}(F(s))$

$$f(t) = L^{-1}\left[\frac{1}{(s+1)(s+2)}\right] = e^{-t} - e^{-2t}$$

Also $f(0) = 0$

$$\therefore \quad L^{-1}[sF(s)] = L^{-1}\left[\frac{s}{(s+1)(s+2)}\right] = f'(t)$$

$$L^{-1}\left(\frac{s}{(s+1)(s+2)}\right) = \frac{d}{dt}\left(e^{-t} - e^{-2t}\right)$$

$$= -e^{-t} + 2e^{-2t}$$

→ Solve the initial value problem

$$y'' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 6$$

Solution. The first step to solve this initial value problem is to take transform of each term in the differential equation.

$$L(y'') + L(4y) = L(0)$$

$$L(y'') + 4L(y) = L(0)$$ [Since Laplace is linear]

Using formula for Laplace transform of function and its derivatives, we get

$$[S^2 Y(s) - s\,y(0) - y'(0)] + 4Y(s) = 0$$

Here $L(y(t)) = Y(s)$ and $L^{-1}(Y(s)) = y(t)$

Put the initial conditions and collect all terms having $Y(s)$ we get.

$$(S^2+4)Y(s) = s+6$$

$$Y(s) = \frac{s+6}{s^2+4} = \frac{s}{s^2+4} + \frac{6}{s^2+4}$$

To get solution, take inverse Laplace on both sides

$$L^{-1}(Y(s)) = L^{-1}\left(\frac{s}{s^2+4}\right) + L^{-1}\left(\frac{6}{s^2+4}\right)$$

$$y(t) = \cos 2t + 3\sin 2t.$$

Example → Solve $y'' + 5y' + 4y = e^{3t}$, $y(0) = 0$, $y'(0) = 3$

Solution → Take Laplace Transform on both sides

$$L(y'') + 5L(y') + 4L(y) = L(e^{3t})$$

$$[S^2Y(s) - s\,y(0) - y'(0)] + 5[sY(s) - y(0)] + 4Y(s) = \frac{1}{s-3}$$

$$(s^2+5s+4)Y(s) = \frac{1}{s-3} + 3$$

$$Y(s) = \frac{1}{(s-3)(s^2+5s+4)} + \frac{3}{(s^2+5s+4)}$$

$$Y(s) = \frac{3s-8}{(s-3)(s+4)(s+1)}$$

$$Y(s) = \frac{A}{s-3} + \frac{B}{s+4} + \frac{C}{s+1}$$

On solving we get $A = \frac{1}{28}$, $B = \frac{-20}{21}$, $C = \frac{11}{12}$

$$\therefore Y(s) = \frac{1}{28(s-3)} - \frac{20}{21(s+4)} + \frac{11}{12(s+1)}$$

(4)

Taking inverse Laplace Transform on both sides, we get

$$L^{-1}(Y(s)) = L^{-1}\left(\frac{1}{28(s-3)}\right) - L^{-1}\left(\frac{20}{21(s+4)}\right) + L^{-1}\left(\frac{11}{12(s+1)}\right)$$

$$y(t) = \frac{1}{28}e^{3t} - \frac{20}{21}e^{-4t} + \frac{11}{12}e^{-t}$$

→ Solve $y'' - 10y' + 9y = 5t$, $y(0) = -1$, $y'(0) = 2$

Solution- $L(y'') - 10L(y') + 9L(y) = 5L(t)$

$$s^2Y(s) - sy(0) - y'(0) - 10(sY(s) - y(0)) + 9Y(s) = \frac{5}{s^2}$$

$$(s^2 - 10s + 9)Y(s) + s - 12 = \frac{5}{s^2}$$

$$Y(s) = \frac{5}{s^2(s^2 - 10s + 9)} + \frac{12 - s}{(s^2 - 10s + 9)}$$

$$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s^2 - 10s + 9)} = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

On solving, we get $A = \frac{50}{81}$, $B = \frac{5}{9}$, $C = \frac{31}{81}$, $D = -2$

$$\therefore Y(s) = \frac{50/81}{s} + \frac{5/9}{s^2} + \frac{31/81}{s-9} - \frac{2}{s-1}$$

Taking inverse laplace Transform on both sides

$$L^{-1}(Y(s)) = \frac{50}{81}L^{-1}\left(\frac{1}{s}\right) + \frac{5}{9}L^{-1}\left(\frac{1}{s^2}\right) + \frac{31}{81}L^{-1}\left(\frac{1}{s-9}\right) - 2L^{-1}\left(\frac{1}{s-1}\right)$$

we get $y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^{t}$

Laplace Transform of Integral

If $L(f(t)) = F(s)$, then $L\left(\int_0^t f(k)\,dk\right) = \frac{F(s)}{s} = \frac{L(f(t))}{s}$

Proof → Let $\phi(t) = \int_0^t f(k)\,dk$

$$\phi'(t) = f(t)$$

Now $L(f(t)) = L(\phi'(t)) = sL(\phi(t)) - \phi(0)$ [Using - (9)]

$\therefore\ L(f(t)) = sL(\phi(t)) \quad [\because \phi(0)=0]$

ie $L(\phi(t)) = \frac{1}{s} L(f(t))$

$$\boxed{L\left(\int_0^t f(k)\,dk\right) = \frac{1}{s} L(f(t)) = \frac{F(s)}{s}} \quad -(13)$$

Remark →

$$\boxed{L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t f(k)\,dk} \quad -(14)$$

where $F(s) = L(f(t))$

e.g. find $L^{-1}\left(\frac{16}{s^3+9s}\right)$

Let $F(s) = \frac{16}{s^2+9}$, Then $f(t) = L^{-1}(F(s))$

$$f(t) = L^{-1}\left(\frac{16}{s^2+9}\right) = \frac{16\sin 3t}{3}$$

$\therefore$ Using (14) $L^{-1}\left(\frac{F(s)}{s}\right) = L^{-1}\left(\frac{16}{s(s^2+9)}\right) = \int_0^t f(k)\,dk$

$$= \int_0^t \frac{16\sin 3k}{3}\,dk$$

$$= -\frac{16}{3}\left[\frac{\cos 3k}{3}\right]_0^t$$

$$= -\frac{16}{3}\left[\frac{\cos 3t}{3} - \frac{1}{3}\right]$$

$$L^{-1}\left(\frac{16}{s^3+9s}\right) = \frac{16}{9} - \frac{16}{9}\cos 3t$$

- Solve $L^{-1}\left(\frac{\omega}{s^2(s^2+\omega^2)}\right)$

Solution → Let $F(s) = \frac{\omega}{s^2+\omega^2}$. Then $f(t) = L^{-1}(F(s))$

$$f(t) = L^{-1}\left(\frac{\omega}{s^2+\omega^2}\right) = \sin\omega t$$

Using (14) $L^{-1}\left(\frac{F(s)}{s}\right) = L^{-1}\left(\frac{\omega}{s(s^2+\omega^2)}\right) = \int_0^t \sin\omega k \, dk$

$$= \left[\cancel{\frac{\sin\omega k}{\omega}} - \frac{\cos\omega k}{\omega}\right]_0^t$$

$$= -\frac{\cos\omega t}{\omega} + \frac{1}{\omega} = \frac{1}{\omega}\left[1-\cos\omega t\right]$$

$$L^{-1}\left(\frac{1}{s^2}F(s)\right) = L^{-1}\left(\frac{\omega}{s^2(s^2+\omega^2)}\right) = \int_0^t \frac{1}{\omega}(1-\cos\omega k)\,dk$$

$$= \frac{1}{\omega}\left[k - \frac{\sin\omega k}{\omega}\right]_0^t = \frac{1}{\omega}\left[t - \frac{\sin\omega t}{\omega}\right]$$

$$\therefore \quad L^{-1}\left(\frac{\omega}{s^2(s^2+\omega^2)}\right) = \frac{1}{\omega}\left[t - \frac{\sin\omega t}{\omega}\right]$$

→ Solve the integro-differential equation

$$y' - y - 6\int_0^t y(k)\,dk = \sin t, \quad y(0) = 2$$

Remark → Integro-differential equation is equation that involves both integrals and derivatives of a function.

$$L(y') - L(y) - 6L\left(\int_0^t y(k)\,dk\right) = L(\sin t)$$

$$sY(s) - y(0) - Y(s) - 6\frac{Y(s)}{s} = \frac{1}{s^2+1}$$

$$\left(s - 1 - \frac{6}{s}\right)Y(s) = \frac{1}{s^2+1} + 2$$

$$(s^2-s-6)Y(s) = \frac{s}{s^2+1} + 2s$$

$$Y(s) = \frac{s}{(s^2+1)(s^2-s-6)} + \frac{2s}{\cancel{(s^2+1)}(s^2-s-6)}$$

$$Y(s) = \frac{2s^3+3s}{(s^2+1)(s^2-s-6)} = \frac{2s^3+3s}{(s^2+1)(s+2)(s-3)}$$

$$Y(s) = \frac{As+B}{s^2+1} + \frac{C}{s-3} + \frac{D}{s+2}$$

On solving, we get $A = -\frac{7}{50}$, $B = -\frac{1}{50}$, $C = \frac{63}{50}$, $D = \frac{22}{25}$

$$\therefore Y(s) = -\frac{7s}{50(s^2+1)} - \frac{1}{50(s^2+1)} + \frac{63}{50(s-3)} + \frac{22}{25(s+2)}$$

Taking inverse Laplace Transform we get

$$y(t) = L^{-1}(Y(s)) = -\frac{7}{50} L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{50} L^{-1}\left(\frac{1}{s^2+1}\right) + \frac{63}{50} L^{-1}\left(\frac{1}{s-3}\right) + \frac{22}{25} L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \frac{-7}{50}\cos t - \frac{1}{50}\sin t + \frac{63}{50}e^{3t} + \frac{22}{25}e^{-2t}$$

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First shifting theorem :-

If $L(f(t)) = f(p)$ then

$L(e^{at} f(t)) = f(p-a)$

Proof :- $L(e^{at} f(t)) = \int_0^\infty e^{-pt} e^{at} f(t) dt$

$$= \int_0^\infty e^{-(p-a)t} f(t) \, dt$$

$$= f(p-a)$$

apply this property we get

(1) $L(e^{at} t^n) = \dfrac{n!}{(p-a)^{n+1}}$

(2) $L(e^{at} \sin bt) = \dfrac{b}{(p-a)^2 + b^2}$

(3) $L(e^{at} \cos bt) = \dfrac{p-a}{(p-a)^2 + b^2}$

(4) $L(e^{at} \sinh bt) = \dfrac{b}{(p-a)^2 - b^2}$

(5) $L(e^{at} \cosh bt) = \dfrac{p-a}{(p-a)^2 - b^2}$

Ex:- (i) Find $L(t^3 e^{-3t})$

$$L(t^3) = \frac{3!}{p^4}$$

$$L(e^{-3t}\, t^3) = \frac{3!}{(p+3)^4}$$ (using first shifting theorm)

(ii) Find Laplace of $e^{-t}\cos t \cos 2t$

Firstly $\cos t \cos 2t = \frac{1}{2}(2\cos t \cos 2t)$

$$= \frac{1}{2}(\cos 3t + \cos t)$$

$$L\left(\frac{1}{2}(\cos 3t + \cos t)\right) = \frac{1}{2}\left[L(\cos 3t) + L(\cos t)\right]$$

$$= \frac{1}{2}\left[\frac{p}{p^2+9} + \frac{p}{p^2+1}\right]$$

By first shifting theorm

$$L(e^{-t}\cos t \cos 2t) = \frac{1}{2}\left[\frac{p+1}{(p+1)^2+9} + \frac{p}{(p\mp 1)^2+1}\right]$$

(Assignment)

(1) Find the laplace of

(a) $e^{t}\, t^{-1/2}$ Ans:- $\frac{\sqrt{\pi}}{\sqrt{p-1}}$

(b) $e^{-t}(\sin 2t - 2t\cos 2t)$ Ans $\frac{16}{(p^2+2p+5)^2}$

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(c) $L(e^{-2t}\sin 4t)$ Ans :- $\frac{4}{p^2+4p+20}$

First shifting property for Inverse Laplace :-

If $L^{-1}(f(p)) = F(t)$

then $L^{-1}(f(p-a)) = e^{at} F(t)$

proof :- $f(p) = \int_0^\infty e^{-pt} f(t)\, dt$

$f(p-a) = \int_0^\infty e^{-(p-a)t} f(t)\, dt$

$= \int_0^\infty e^{-(pt)} e^{at} F(t)\, dt$

$= L(e^{at} F(t))$

$\therefore L^{-1}(f(p-a)) = e^{at} F(t)$

eg :- Find Inverse laplace of $\frac{1}{\sqrt{2p+3}}$

$L^{-1}\left(\frac{1}{\sqrt{2p+3}}\right) = L^{-1}\left(\frac{1}{\sqrt{2}\left(\sqrt{p+3/2}\right)}\right)$

$= \frac{1}{\sqrt{2}} L^{-1}\left(\frac{1}{\sqrt{p+3/2}}\right)$

$$= \frac{1}{\sqrt{2}} e^{-3/2 t} L^{-1}\left(\frac{1}{\sqrt{p}}\right)$$

[by first shifting theorem]

$$= \frac{1}{\sqrt{2}} e^{-3/2 t} \frac{1}{\sqrt{\pi t}}$$

$$= \frac{1}{\sqrt{2\pi t}} e^{-3/2 t}$$

(2) Find $L^{-1} \frac{1+2p}{(p+2)^2(p-1)^2}$

Sol:- $\frac{1+2p}{(p+2)^2(p-1)^2} = \frac{A}{p+2} + \frac{B}{(p+2)^2} + \frac{C}{p-1} + \frac{D}{(p-1)^2}$

$$1+2p = A(p+2)(p-1)^2 + B(p-1)^2 + C(p-1)(p+2)^2 + D(p+2)^2$$

$$= A(p^3-3p+2) + B(p^2-2p+1) + C(p^3+3p^2-4) + D(p^2+4p+4)$$

Comparison gives

$A+C=0$, $B+3C+D=0$, $-3A-2B+4D=2$

$2A+B-4C+4D=1$

Solving we get $A=0$, $C=0$, $B=\frac{-1}{3}$

$D=\frac{1}{3}$

$$\therefore \frac{1+2p}{(p+2)^2(p-1)^2} = \frac{-1}{3(p+2)^2} + \frac{1}{3(p-1)^2}$$

$$L^{-1}\left(\frac{1+2p}{(p+2)^2(p-1)^2}\right) = -\frac{1}{3} L^{-1}\left(\frac{1}{(p+2)^2}\right) + \frac{1}{3} L^{-1}\left(\frac{1}{(p-1)^2}\right)$$

$$= -\frac{1}{3} e^{-2t} t + \frac{1}{3} e^{t} t$$

$$= \frac{t}{3}\left(e^{t} - e^{-2t}\right)$$

Unit step function :-

(or Heaviside's Unit step function)

The unit Step function $U(t-a)$ is defined as

$$U(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t \geq a \end{cases} \quad \text{where } a \geq 0$$

As a particular case

$$U(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

The Product $F(t)\,U(t-a) = \begin{cases} 0 & \text{for } t < a \\ F(t) & \text{for } t \geq a \end{cases}$

Laplace of Unit Step function

$$L(U(t-a)) = \int_0^\infty e^{-pt}\, U(t-a)\, dt$$

$$= \int_0^a e^{-pt} \cdot 0\, dt + \int_a^\infty e^{-pt} \cdot 1\, dt$$

$$= 0 + \left[\frac{e^{-pt}}{-p} \right]_a^\infty = \frac{1}{p} e^{-ap}$$

In particular $L(U(t)) = \frac{1}{p}$

Laplace of Unit Step function

$$L(U(t-a)) = \int_0^\infty e^{-pt} U(t-a)\,dt$$

$$= \int_0^a e^{-pt}\cdot 0\,dt + \int_a^\infty e^{-pt}\cdot 1\,dt$$

$$= 0 + \left[\frac{e^{-pt}}{-p}\right]_a^\infty = \frac{1}{p} e^{-ap}$$

In particular $L(U(t)) = \frac{1}{p}$

<u>second shifting theorm :-</u>

If $L(f(t)) = f(p)$ then $L\{F(t-a)U(t-a)\} = e^{-ap} f(p)$

proof :- $L(F(t-a)U(t-a)) = \int_0^\infty e^{-pt} F(t-a) U(t-a)\,dt$

$$= \int_a^\infty e^{-pt} F(t-a)\,dt$$

[by def of $U(t-a)$]

$$= \int_0^\infty e^{-p(U+a)} F(U)\,dU$$

where $U = t-a$

$$= e^{-ap} \int_0^\infty e^{-pU} F(U)\,dU$$

$$= e^{-ap} f(p)$$

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Q:- Find the laplace transforms of

$e^{-3t} \, U(t-2)$

Sol:- $e^{-3t} = e^{-3((t-2)+2)}$

$= e^{-6} e^{-3(t-2)}$

$e^{-3t} U(t-2) = e^{-6} e^{-3(t-2)} U(t-2)$

Now comparing with $F(t-a)\,U(t-a)$
we get $a = 2$ and $F(t) = e^{-3t}$

$$f(p) = L(F(t)) = \frac{1}{p+3}$$

$$\therefore L(e^{-3t} U(t-2)) = e^{-6} L(e^{-3(t-2)} U(t-2))$$

$$= e^{-6} e^{-2p} f(p)$$

$$= \frac{e^{-2(p+3)}}{p+3}$$

Q:- Find Laplace transforms of

$\sin t \; U(t-\pi)$

Sol:- $\sin t = \sin[(t-\pi) + \pi]$

$= -\sin(t-\pi)$

Comparing $-\sin(t-\pi)\,U(t-\pi)$
with $F(t-a)\,U(t-a)$ we get

$a = \pi$ and $F(t) = -\sin t$

$f(p) = L(F(t)) = -\frac{1}{p^2+1}$

$\therefore L(\sin t\, u(t-\pi)) = L(-\sin(t-\pi)\, u(t-\pi))$

$= e^{-\pi p} f(p)$

$= -\frac{e^{-\pi p}}{p^2+1}$ [by Second Shifting]

Second Shifting for Inverse Laplace :-

If $L^{-1}(f(p)) = F(t)$ then

$L^{-1}(e^{-ap} f(p)) = G(t)$

where $G(t) = \begin{cases} F(t-a) & t>a \\ 0 & t<a \end{cases}$

eg :- $L^{-1}\left(\frac{p+8}{p^2+4p+5}\right)$.

$L^{-1}\left(\frac{(p+2)+6}{(p+2)^2+1}\right) = L^{-1}\left(\frac{p+2}{(p+2)^2+1}\right) + 6L^{-1}\left(\frac{1}{(p+2)^2+1}\right)$

$= e^{-2t} L^{-1}\left(\frac{p}{p^2+1}\right) + 6e^{-2t} L^{-1}\left(\frac{1}{p^2+1}\right)$

using first shifting

$= e^{-2t}\cos t + 6e^{-2t}\sin t$

Q:- evaluate $L^{-1}\left(\frac{e^{-2p}}{p^2}\right)$

Sol:- $L^{-1}\left(\frac{1}{p^2}\right) = t = F(t)$

$\therefore L^{-1}\left(e^{-2p}\frac{1}{p^2}\right) = \begin{cases} t-2 & t>2 \\ 0 & t<2 \end{cases}$

$= (t-2)U(t-2)$

Q:-2 $L^{-1}\left(\frac{pe^{-p/2}+\pi e^{-p}}{p^2+\pi^2}\right)$

Sol:- $L^{-1}\left(\frac{p}{p^2+\pi^2}\right) = \cos \pi t$

and $L^{-1}\left(\frac{\pi}{p^2+\pi^2}\right) = \sin \pi t$

$\therefore L^{-1}\left(\frac{pe^{-p/2}}{p^2+\pi^2}\right) = \cos \pi\left(t-\frac{1}{2}\right)U\left(t-\frac{1}{2}\right)$

[by second Shifting thm]

$= \sin \pi t \, U\left(t-\frac{1}{2}\right)$

and $L^{-1}\left(\frac{\pi e^{-p}}{p^2+\pi^2}\right) = \sin\pi(t-1)U(t-1)$

$= -\sin\pi t \, U(t-1)$

$\therefore L^{-1}\left(\frac{pe^{-p/2}+\pi e^{-p}}{p^2+\pi^2}\right) = \sin\pi t\left[U\left(t-\frac{1}{2}\right)-U(t-1)\right]$

Assignment :-

(I) Find $L^{-1}\left(\frac{e^{-p}}{\sqrt{p+1}}\right)$

Ans $\frac{e^{-(t-1)}}{\sqrt{\pi(t-1)}} u(t-1)$

(II) $L^{-1}\left(\frac{p e^{-(2\pi/3)p}}{p^2+9}\right)$

Ans :- $\cos 3t \, u\left(t-\frac{2\pi}{3}\right)$

(III) $L\left(e^{-t}(1-u(t-2))\right)$

A = $\frac{1-e^{-2(1+p)}}{1+p}$

(IV) $L(t^2 u(t-3))$

A $e^{-3p}\left(\frac{2+6p+9p^2}{p^3}\right)$

V $L^{-1}\left(\frac{15}{p^2+4p+13}\right)$

A $5e^{-2t}\sin 3t$

VI $L^{-1}\left(\frac{3p+1}{(p+1)^4}\right)$

A = $e^{-t}\left(\frac{3}{2}t^2-\frac{1}{3}t^3\right)$

Probability Distribution

Trial and event :- Let an experiment be repeated under essentially the same conditions and let it result in any one of the several possible outcomes. Then the experiment is called a trial and the possible outcomes are known as events or cases.

For example :-

(1) Tossing of a coin is a trial and the turning up of tail is an event.

Exhaustive Events :- The total number of all possible outcomes in any trial is known as exhaustive events or exhaustive cases. If $A_1, A_2, A_3, ---, A_n$ are exhaustive events then $A_1 \cup A_2 \cup A_3 \cup --- \cup A_n = S$, the sample space.

for eg :- In tossing a coin, there are two exhaustive cases head and trail.

Favourable events :- The cases which entail the happening of an event are said to be favourable to the event. It is the total number of possible outcomes in which the specified event happens.

For eg :- In a throw of two dice, the number of cases favourable to getting a sum 6 is 5 viz. (1,5), (5,1), (2,4), (4,2), (3,3).

Mutually exclusive events :- Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes the happening of all others i.e If no two or more than two of them can happen simultaneously in the same trial. If $A_1, A_2, \ldots, A_n$ are mutually exclusive events then $A_1 \cap A_2 \cap \ldots \cap A_n = \phi$

Independent and dependent Events :-

Two or more events are said to be independent if the happening or non-happening of any one does not depend by the happening or non-happening of any other. Otherwise they are said to be dependent.

Random Experiment :-

Occurrences which can be repeated a number of times, essentially under the same conditions, and whose result cannot be predicated before hand are known as random experiments.
For example, rolling a die, tossing a coin, taking out balls from an urn.

Sample Space :- Out of the several possib

Outcomes of a random experiment. One an
Only one can take place in a trail. The set of all these possible outcomes is c the sample space for the particula experiment and is denoted by S.
For example, if a coin is tossed th

possible outcomes are H (Head) and T(Tail)
Thus $S = \{H, T\}$

Sample point :- The elements of S, the sample S are called sample points.
For example if a coin is tossed and H and T denote "Head" and Trail respectively then $S = \{H, T\}$ The two sample points are H and T.

Event :- Every subset of S, the sample is called an event.
Since $S \subset S$ itself is an event called a certain event.
Also $\phi \subset S$ the null set is also an event called an impossible event el
If $e \in S$ then e is called an event
Every elementary event contains only one sample.

Random Variable :-

If the numerical values assumed by a variable are the result of some chance factors so that a particular value cannot be exactly predicted in advance the variable is then called a random variable.

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Continous Random Variable :-

A continous random variable is one which can assume any value within an interval i.e all values of a continous scale.
For eg :- weights (in kg) of a group of Individuals.

Discrete Random Variable :-

A discrete random variable is one which can assume only isolated values. For e.g.
the number of heads in 4 tosses of a coin is a discrete random variable as it cannot assume value other than 0, 1, 2, 3, 4

Discrete Probability Distribution

Let a random X assume values $x_1, x_2, x_3, \ldots, x_n$ with probabilites $p_1, p_2, \ldots, p_n$ respectively where $P(X = x_i) = p_i \geq 0$ for each x_i and $p_1 + p_2 + \ldots + p_n = \sum_{i=1}^{n} p_i = 1$ Then

X :	x_1	x_2	x_3	-----	x_n
$P(X)$:	P_1	P_2	P_3	----	P_n

is called discrete probability distribution for X and it spells out how a total probability of 1 is distributed over several values of the random variable.

Mean and Variance :-

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For discrete distribution :-

$$\text{Mean} = \mu = \sum p_i x_i$$

$$\text{Variance} = \sigma^2 = \sum p_i x_i^2 - \mu^2$$

For continous distribution :-

$$\text{Mean} = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\text{Variance} = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$